MAC 2311
Section 4.1
max / min

An absolute max is the largest maximum for the graph in question.

An absolute min is the smallest minimum for the graph in question.

technical definitions

Let c be any number in the domain D of a function $f(x)$

c is an absolute maximum value of f on D if

$f(c) \geq f(x)$ for all x in D.

c is an absolute minimum value of f on D if

$f(c) \leq f(x)$ for all x in D.

NOTE For a closed interval, the endpoints can be potential maximums or minimums.

relative maximum (local) : A max for a given interval

local max if $f(c) \geq f(x)$ when x is near c

relative minimum : A min for a given interval

local min if $f(c) \leq f(x)$ when x is near c

to test for max/mins, we use the first derivative analysis.

<u>Summary</u>

1. find $f'(x)$
2. set $f'(x) = 0$ and solve for x
3. determine for what x values $f'(x)$ is undefined.
4. analyze with test points
5. Be sure to include endpoints if the domain is finite.

Indications

Let c = critical number (possible max/min)

f': + | − ; inc. (c) decr. — max

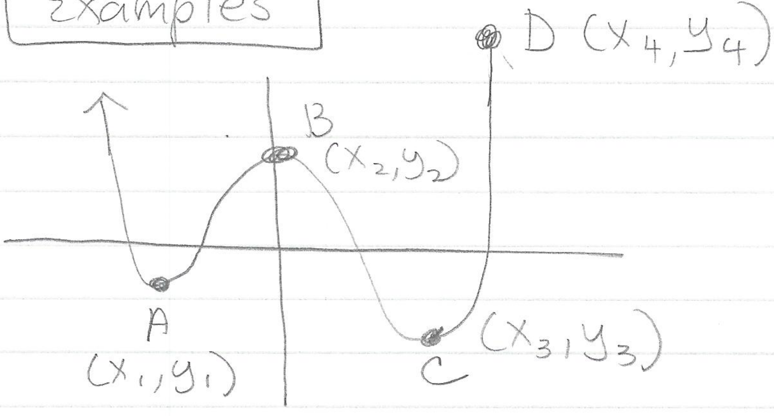
f': − | + ; (c) — min

f': + | + ; inc (c) inc

f': − | − ; dec (c) dec

$\boxed{\text{possible}}\ \boxed{\text{inflection points}}$
(another test — the second derivative analysis — is needed to confirm this)

Examples



$\boxed{\text{local max}}$ B & D

occurs at x_2
the value is $f(x_2) = y_2$

occurs at x_4
the value is $f(x_4) = y_4$

$\boxed{\text{absolute max}}$ none since left tail approaches ∞

$\boxed{\text{local mins}}$ occur at A & C

$\boxed{A}$ occurs at x_1
value is $f(x_1) = y_1$

$\boxed{C}$ occurs at x_3
value is $f(x_3) = y_3$

$\boxed{\text{absolute min}}$ lowest of all mins $\Rightarrow$ $\boxed{\text{at } C}$

Fermat's Theorem

If f has a local max or min at c, then $f'(c)$ exists and $f'(c) = 0$.

$\boxed{\text{Examples}}$ $\boxed{\text{Concentrate on \#29–44, 47–62}}$

$\boxed{\text{We will graph in a later section}}$

32 Find the critical numbers

$\boxed{\text{critial numbers}}$ x values when $\boxed{f'(x)=0}$ & $\boxed{f'(x) \text{ is undefined.}}$

$\boxed{\text{stationary points}}$ $\boxed{\text{singular points}}$

$$f(x) = 2x^3 + x^2 + 2x$$
$$f'(x) = 6x^2 + 2x + 2$$

$\boxed{f'(x) = 0}$

$$6x^2 + 2x + 2 = 0$$
$$2(3x^2 + x + 1) = 0$$

$\boxed{f'(x) \text{ undefined}}$

none

$$x = \frac{-1 \pm \sqrt{1 - 4(3)(1)}}{2(3)} = \frac{-1 \pm \sqrt{-11}}{6} = \frac{-1 \pm \sqrt{11}\, i}{6}$$

(not real)
so no critical numbers.

so the graph of
$f(x) = 2x^3 + x^2 + 2x$
has no max or mins.

36 $h(p) = \dfrac{p-1}{p^2+4}$

$h'(p) = \dfrac{1(p^2+4) - (2p)(p-1)}{(p^2+4)^2}$ $\quad f = p-1 \quad g = p^2+4$

(Quotient) $\quad f' = 1 \quad g' = 2p$

$h'(p) = \dfrac{p^2+4-2p^2+2p}{(p^2+4)^2} = \dfrac{-p^2+2p+4}{(p^2+4)^2}$

$\boxed{h'(p) = 0}$

(num $= 0$)

$-p^2+2p+4 = 0$

$-(p^2-2p-4) = 0$

$p = \dfrac{+2 \pm \sqrt{4-4(1)(-4)}}{2(1)}$

$p = \dfrac{2 \pm \sqrt{20}}{2} = \dfrac{2 \pm 2\sqrt{5}}{2}$

$p = 1 \pm \sqrt{5} \approx 3.24$ & -1.24

$\boxed{h'(p) \text{ undefined}}$

(denom $= 0$)

$p^2+4 = 0$

$p^2 = -4$

$p = \pm 2i$

not real

$\Rightarrow$ no critical numbers.

p′ $\quad - \quad | \quad + \quad | \quad -$

-1.24 (min) $\quad 3.24$ (max)

$\boxed{\text{test}}$ using p′

$p = -2 \quad \dfrac{-(-2)^2+2(-2)+4}{((-2)^2+4)^2} = \dfrac{-4-4+4}{16^2} = -$

$p = 0 \quad \dfrac{-(0)^2+2(0)+4}{(0^2+4)^2} = \dfrac{4}{16} = +$

$p = 4 \quad \dfrac{-(4)^2+2(4)+4}{(4^2+4)^2} = \dfrac{-16+8+4}{(20)^2} = -$

$\boxed{47-62}$ Absolute max/min $\rightarrow$ include endpoints.

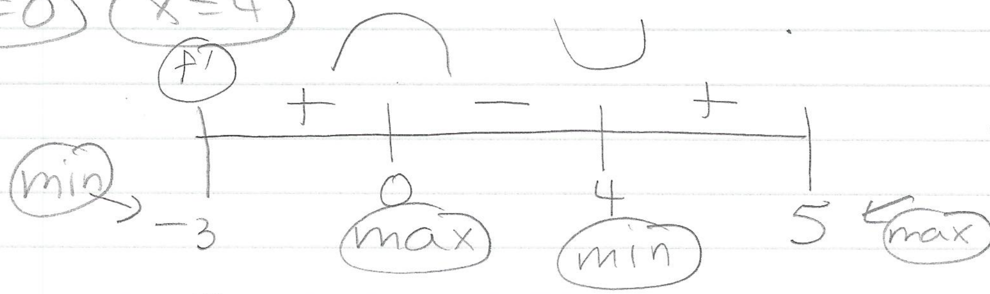
50 $f(x) = x^3 - 6x^2 + 5 \quad [-3, 5]$

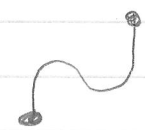
$f(x) = x^3 - 6x^2 + 5$
$f'(x) = 3x^2 - 12x$

$\boxed{f'(x) = 0}$ $\boxed{f'(x) \text{ undefined}}$

$3x^2 - 12x = 0$
$3x(x-4) = 0$
$x = 0 \quad x = 4$

none $\rightarrow$ no denominator





basic shape

$\boxed{\text{test points}}$ into $f'(x) = 3x^2 - 12x$

$x = -2 \quad 3(-2)^2 - 12(-2) = 12 + 24 \quad +$
$x = 1 \quad 3(1)^2 - 12(1) = -9 \quad -$
$x = 4.5 \quad 3(4.5)^2 - 12(4.5) = 6.75 \quad +$

$\boxed{\text{use } f(x) = x^3 - 6x^2 + 5}$ to find point values.

$\boxed{\text{min}}$
at $x = -3 \quad f(-3) = (-3)^3 - 6(-3)^2 + 5 = -76$
$x = 4 \quad f(4) = (4)^3 - 6(4)^2 + 5 = -27$
absolute min $\rightarrow$ occurs at $x = -3$
value $y = -76$

$\boxed{\text{max}}$
at $x = 0 \quad f(0) = (0)^3 - 6(0)^2 + 5 = 5$
$x = 5 \quad f(5) = (5)^3 - 6(5)^2 + 5 = -20$
absolute max $\rightarrow$ occurs when $x = 0$
value $y = 5$

56 $\boxed{f(t) = \dfrac{\sqrt{t}}{1+t^2}}$ $[0, 2]$

$$f'(t) = \frac{\frac{1}{2}t^{-1/2}(1+t^2) - t^{1/2}(2t)}{(1+t^2)^2}$$

$f = t^{1/2}$ $g = 1+t^2$
$f' = \frac{1}{2}t^{-1/2}$ $g' = 2t$

$$f'(t) = \frac{\frac{1+t^2}{2t^{1/2}} - 2t^{3/2}}{(1+t^2)^2} \cdot \frac{2t^{1/2}}{2t^{1/2}}$$

$$f'(t) = \frac{1+t^2 - 2t^{3/2}(2t^{1/2})}{(1+t^2)^2(2t^{1/2})} = \frac{1+t^2 - 4t^2}{(1+t^2)^2(2t^{1/2})}$$

$$\boxed{f'(t) = \frac{1-3t^2}{(1+t^2)^2(2t^{1/2})}}$$

$\boxed{f'(t) = 0}$

(num $= 0$)

$1 - 3t^2 = 0$

$-3t^2 = -1$

$t^2 = \frac{1}{3}$

$t = \pm\sqrt{\frac{1}{3}}$

$t \simeq \pm 0.58$

$\boxed{f'(t) \text{ undef}}$

(denom $= 0$)

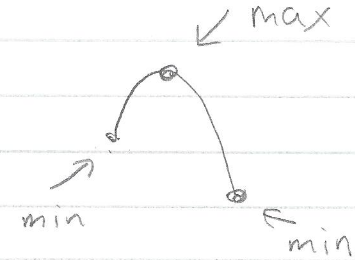
$(1+t^2)^2 = 0$ $(2t^{1/2}) = 0$

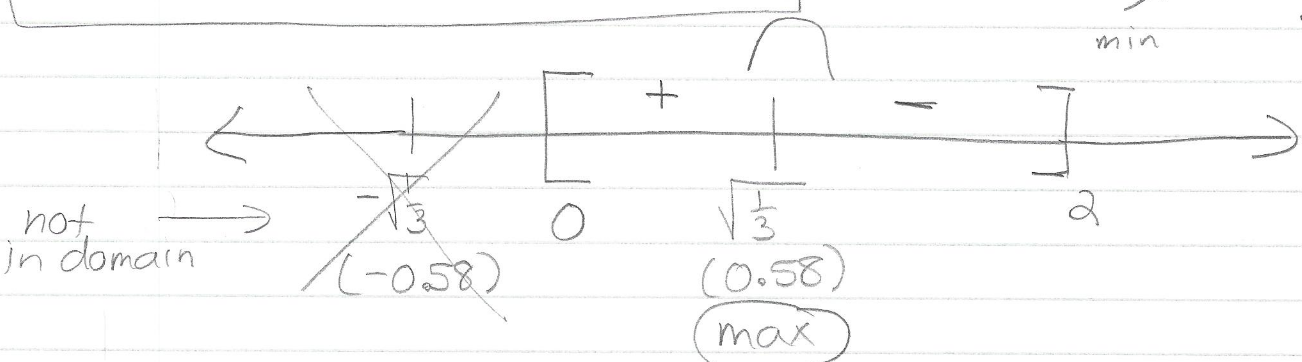
$1+t^2 = 0$ $t = 0$

$\cancel{t^2 = -1}$

56 continued $\qquad [0, 2]$

$$\boxed{\text{using } f'(t) = \frac{1-3t^2}{(1+t^2)^2(2t^{1/2})}}$$





test points

$$f'(1) = \frac{1-3(+1)^2}{(1+(+1)^2)^2(2(+1)^{1/2})} = \frac{1-3}{4(2)(1)} = \frac{-2}{8} = (-)$$

$$f'(0.5) = \frac{1-3(0.5)^2}{(1+(0.5)^2)^2(2(0.5)^{1/2})} = \frac{0.25}{(+)} = (+)$$

now use $f(t) = \frac{\sqrt{t}}{1+t^2}$ to find the point values

$\boxed{\text{max}}$ one possibility (local & absolute max)

$$t = \sqrt{\tfrac{1}{3}} \qquad f\left(\sqrt{\tfrac{1}{3}}\right) = \frac{\sqrt{\sqrt{\frac{1}{3}}}}{1+\left(\sqrt{\frac{1}{3}}\right)^2} \simeq \frac{3^{3/4}}{4} \simeq 0.570$$

$\boxed{\text{mins}}$

$$t = 0 \qquad f(0) = \frac{\sqrt{0}}{1+0^2} = 0 \longleftarrow \boxed{\text{absolute min}}$$

$$t = 2 \qquad f(2) = \frac{\sqrt{2}}{1+2^2} \simeq 0.283$$

Hints

1. It is fine to use decimal approximations.
2. It is also a good idea to program the original function and the first derivate into the graphing calculator. That way, the calculator can help with the evaluation phase.
3. check your derivatives using wolfram alpha or another online program before you proceed. If your derivative is off, your interpretation will also be off.